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High-temperature expansions for the strongly correlated Hubbard model in the limit of infinite dimension

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Abstract. High-temperature expansions for the strongly correlated Hubbard model in the limit of infinite spatial dimension $d \to \infty$, with hopping scaled by $d^{-1/2}$, are developed for the Gibbs free energy and susceptibility to order T^{-10} for arbitrary values of the density. From Padé and other analysis we find no evidence of a phase transition at a finite critical temperature for any value of the density.

1. Introduction

The Hubbard model [1] has attracted much interest over the years as a possible model of itinerant ferromagnetism [2–4], and, more recently, in connection with theories of high-temperature superconductivity [5–7]. The model is described by the Hamiltonian

$$\mathcal{H} = -t \sum_{(i,j)} \sum_{\sigma=\uparrow,\downarrow} \left(a_{i,\sigma}^{\dagger} a_{j,\sigma} + a_{j,\sigma}^{\dagger} a_{i,\sigma}\right) + U \sum_{i=1}^{N} n_{i,\uparrow} n_{i,\downarrow} - h \sum_{i=1}^{N} \left(n_{i,\uparrow} - n_{i,\downarrow}\right) \tag{1.1}$$

where (ij) denotes nearest-neighbour lattice sites, $a_{i,\sigma}$ $(a_{i,\sigma}^{\dagger})$ is the annihilation (creation) operator for an electron on site i with spin $\sigma = \uparrow, \downarrow$ and $n_{i,\sigma} = a_{i,\sigma}^{\dagger} a_{i,\sigma}$ is the corresponding number operator. The first term in (1.1) represents the kinetic energy, with nearest-neighbour hopping energy t, the second term represents an on-site repulsion with energy U, and the last term represents the interaction of the electron's spin with an external magnetic field h.

In spite of much work, there are very few rigorous results for the Hubbard model. Certain ground state properties are known exactly for the one-dimensional model [8–10] and in higher dimensions it is known [11, 12] that (1.1), with the number of electrons equal to $N-1$, has a ferromagnetic ground state. On the basis of these results, and various approximate methods, such as the Hartree–Fock [13] and Gutzwiller variational [14] methods, it is widely believed that the Hubbard model (1.1) has a ferromagnetic ground state, at least near half-filling density. There are no rigorous results

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and there is no common agreement, however, regarding the existence of a ferromagnetic state, or phase transition, at a finite temperature.

The method of high-temperature series expansions, which has proved so successful in the study of critical phenomena in classical lattice systems [15], has been applied to the Hubbard model, but so far with limited success and inconclusive and contradictory results. Plischke [16] for example, did not take proper account of the anti-commutation properties of the Fermi operators, while the series of Brauneck [17], Kubo [18] and Kubo and Tada [19], which corrected this error, were too short and too irregular for reliable conclusions to be reached. The series of Kubo [18] were also observed to contain errors [20].

Our purpose here is to report on further series work for the strongly correlated Hubbard model in which $U \to \infty$, i.e. double occupancy of sites by electrons of either spin is prohibited. In this limit, the Hamiltonian (1.1) is replaced by [21]

$$\tilde{\mathcal{H}} = -t\sum_{(i,j)}\sum_{\sigma=\uparrow,\downarrow}\left(\tilde{a}^{\dagger}_{i,\sigma}\tilde{a}_{j,\sigma} + \tilde{a}^{\dagger}_{j,\sigma}\tilde{a}_{i,\sigma}\right) - h\sum_{i=1}^{N}\left(n_{i,\uparrow} - n_{i,\downarrow}\right) \tag{1.2}$$

where

$$\tilde{a}_{i,\sigma} = a_{i,\sigma}\left(1 - n_{i,-\sigma}\right) \tag{1.3}$$

and the problem is to compute the grand canonical partition function

$$\begin{aligned} Z_{\mathrm{G}} &= \lim_{U\to\infty} \mathrm{Tr}\left\{\exp\left[-\beta\left(\mathcal{H} - \mu\mathcal{N}\right)\right]\right\} \\ &= \mathrm{Tr}\left[\mathcal{P}\prod_{\sigma=\uparrow,\downarrow} z_{\sigma}^{\mathcal{N}_{\sigma}} \exp\left(\tau\mathcal{H}_0\right)\mathcal{P}\right] \end{aligned} \tag{1.4}$$

where $\tau = \beta t$,

$$\mathcal{N}_{\sigma} = \sum_{i=1}^{N} n_{i,\sigma} \qquad \mathcal{N} = \mathcal{N}_{\uparrow} + \mathcal{N}_{\downarrow} \tag{1.5}$$

$$z_{\uparrow} = \exp[\beta(\mu + h)] \qquad z_{\downarrow} = \exp[\beta(\mu - h)] \tag{1.6}$$

$$\mathcal{H}_0 = \sum_{(i,j)}\sum_{\sigma=\uparrow,\downarrow}\left(\tilde{a}^{\dagger}_{i,\sigma}\tilde{a}_{j,\sigma} + \tilde{a}^{\dagger}_{j,\sigma}\tilde{a}_{i,\sigma}\right) \tag{1.7}$$

and the operator

$$\mathcal{P} = \prod_{i=1}^{N}\left(1 - n_{i,\uparrow}n_{i,\downarrow}\right) \tag{1.8}$$

projects out doubly occupied states in the trace, taken over all states, in (1.4).

In the following section, we formulate the general problem of generating high-temperature expansions, in power of τ, for the free energy and magnetic susceptibility for the strongly correlated model. As a check on our procedure, we recalculated the square lattice series to order τ^8 and found agreement with Kubo and Tada.

Instead of further reproduction and extension of existing series in two and three dimensions we concentrated our attention on the infinite-dimensional limit of the strongly correlated model. In order to obtain non-trivial results we need to re-scale the hopping term by $d^{-1/2}$ before taking the limit of lattice dimensionality $d \to \infty$ [22]. In the high-temperature series expansions for this limiting model, only maximally extended graphs contribute, so that the complexity of the problem is considerably reduced. Indeed, some exact results are known for this limiting model [23, 24], and there is some hope of an exact mean-field-type solution.

We found, however, that due mainly to the anti-commutativity of the Fermi operators, the generation of high-temperature series expansions for even this simplest limiting Hubbard model was still extremely complicated, and that as a result, computer time limitations restricted us to generating series only up to order τ^{10}.

As in the finite-dimensional case, the series we obtained were quite irregular and difficult to analyse by simple extrapolation techniques. A simple plot of the susceptibility χ^{-1} computed from the series, against τ^{-1}, however, leads us to believe that the limiting model does not have a phase transition at a finite temperature. This conclusion is supported by Padé analysis [25] and other analyses and is consistent with recent rigorous results [26] for the related Falicov–Kimball model in the infinite-dimensional limit.

Finally, as a check on the validity of our reciprocal susceptibility plot, we repeated the calculation for our square lattice series and reached the expected conclusion that the strongly correlated two-dimensional model does not have a phase transition.

As conventional wisdom dictated that phase transitions become more likely with increasing dimension, we conjecture that the strongly correlated Hubbard model does not have a phase transition at a finite temperature in any dimension and for any density.

2. High-temperature expansions

The simplest and most straightforward way to generate high-temperature expansions is to expand the exponential in (1.4) in powers of $\tau = \beta t$, to obtain

$$Z_{\mathrm{G}} = Z_0 \left\{ 1 + \sum_{n=1}^{\infty} \frac{\tau^n}{n!} \left\langle \left[\sum_{\langle i,j \rangle,\sigma} \left(\tilde{a}^{\dagger}_{i,\sigma} \tilde{a}_{j,\sigma} + \tilde{a}^{\dagger}_{j,\sigma} \tilde{a}_{i,\sigma} \right) \right]^n \right\rangle_0 \right\} \tag{2.1}$$

where the 'unperturbed' grand-canonical partition function Z_0 is given by

$$Z_0 = \mathrm{Tr} \left(\mathcal{P} \prod_{\sigma=\uparrow,\downarrow} z_\sigma^{\mathcal{N}_\sigma} \mathcal{P} \right) = \left(1 + z_\uparrow + z_\downarrow \right)^N \tag{2.2}$$

and the expectation value in (2.1) with respect to the unperturbed system is defined by

$$\langle A \rangle_0 \equiv Z_0^{-1} \mathrm{Tr} \left(A \mathcal{P} \prod_{\sigma=\uparrow,\downarrow} z_\sigma^{\mathcal{N}_\sigma} \mathcal{P} \right). \tag{2.3}$$

In order to evaluate the expectation value appearing in (2.1) we expand the sum raised to the power n and associate with each term $\tilde{a}^{\dagger}_{i,\sigma} \tilde{a}_{j,\sigma}$ a 'particle' with spin

σ moving from site j to a nearest-neighbour site i. In this way, every term in the expanded sum is uniquely associated with an n-step movement of some number of particles on the lattice.

For example, when $n = 4$, the terms

$$\tilde{a}^{\dagger}_{1,\uparrow}\tilde{a}_{4,\uparrow}\tilde{a}^{\dagger}_{3,\uparrow}\tilde{a}_{2,\uparrow}\tilde{a}^{\dagger}_{4,\uparrow}\tilde{a}_{3,\uparrow}\tilde{a}^{\dagger}_{2,\uparrow}\tilde{a}_{1,\uparrow} \tag{2.4}$$

$$\tilde{a}^{\dagger}_{3,\downarrow}\tilde{a}_{1,\downarrow}\tilde{a}^{\dagger}_{1,\downarrow}\tilde{a}_{3,\downarrow}\tilde{a}^{\dagger}_{2,\uparrow}\tilde{a}_{1,\uparrow}\tilde{a}^{\dagger}_{1,\uparrow}\tilde{a}_{2,\uparrow} \tag{2.5}$$

and

$$\tilde{a}^{\dagger}_{3,\downarrow}\tilde{a}_{4,\downarrow}\tilde{a}^{\dagger}_{4,\downarrow}\tilde{a}_{3,\downarrow}\tilde{a}^{\dagger}_{1,\uparrow}\tilde{a}_{2,\uparrow}\tilde{a}^{\dagger}_{2,\uparrow}\tilde{a}_{1,\uparrow} \tag{2.6}$$

are associated with the movements shown in figures 1(a), (b) and (c) respectively.

Notice that, in general, from (2.3), the only movements which contribute to the expectation value in (2.1) are those with the same initial and final particle configurations which have no intermediate doubly occupied sites. The situations shown in figure 1 are examples of such allowed movements.

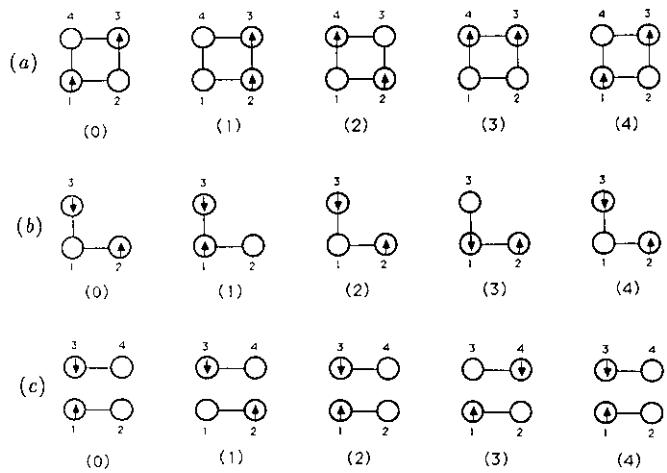

Figure 1. Graphical representation of the terms (2.4), (2.5) and (2.6). (↑) and (↓) denote the sites occupied by a spin up and spin down particle respectively and ○ denotes an unoccupied site. The final configuration (4) is reached from the initial configuration (0) through three intermediate particle movements.

To find the expectation value or weight of each allowed movement, we first use the Fermi anti-commutation relations to group together the operators which act on the same lattice site. The sign of the weight is then easily seen to be plus (minus) if the total number of required commutations or transpositions is even (odd). Finally, it is easily checked that the magnitude of the weight is the product of the weights per site which are equal to

$$p_\sigma = z_\sigma \left(1 + z_\uparrow + z_\downarrow\right)^{-1} \tag{2.7}$$

if the annihilation operator for spin σ is to the right of the creation operator for spin state σ at that site, and equal to

$$q = 1 - p_\uparrow - p_\downarrow = \left(1 + z_\uparrow + z_\downarrow\right)^{-1} \tag{2.8}$$

if the creation operator is to the right of the annihilation operator.

For example, the weight of the terms (2.4), (2.5) and (2.6) are respectively $-p_\uparrow^2 q^2$, $p_\uparrow p_\downarrow q$ and $p_\uparrow p_\downarrow q^2$. In general, the magnitude of a weight is easily seen to be the product of a p_σ to the power of the number of spin σ particles in the corresponding movement and q to the power of the number of holes. The sign of the weight, however, must, in general, be determined by actually performing the commutations as described above. An exception is the case where the particles move on a polygon, such as the square in figure 1(a). In such a case, the sign is plus (minus) for movements involving an odd (even) number of particles.

Many particle movements can, of course, take place on underlying bare graphs, such as the square, chain of length 2 and disconnected bonds in figure 1, so that in order to calculate the total contribution arising from a given graph, one must multiply the number of ways of embedding the bare graph in the lattice by the sum of the number of allowed particle movements for a given number of spin up and/or spin down particles on the graph multiplied by their respective weights.

For example, there are $6N$ ways of embedding a chain of length 2 in the square lattice of N sites, four allowed ways of moving one particle, with spin up or spin down, and four allowed ways of moving two particles, with spins independently up or down, giving rise to a total contribution of

$$6N \times 4 \times \left\{ p_\uparrow q^2 + p_\downarrow q^2 + p_\uparrow^2 q + 2p_\uparrow p_\downarrow q + p_\downarrow^2 q \right\} . \tag{2.9}$$

On the other hand, there are $N(N-7)$ ways of embedding two disconnected bonds on the square lattice, six ways of assigning four nearest-neighbour labels to the two bonds and independently one particle of either spin and two movements on each bond giving a total contribution of

$$N(N-7) \times 6 \times \left(2p_\uparrow q + 2p_\downarrow q\right)^2 . \tag{2.10}$$

In general, the weight of a disconnected graph can be obtained from the weight of its component graphs. Thus if a disconnected graph G_{ab} is composed of two subgraphs G_a and G_b (which may be connected or disconnected), the weight of an N_{ab}-step movement on G_{ab} is given by

$$W_{ab}\left(N_{ab}\right) = \sum_{N_a+N_b=N_{ab}} \frac{N_{ab}!}{N_a! N_b!} W_a\left(N_a\right) W_b\left(N_b\right) \tag{2.11}$$

where $W_a\left(N_a\right)$ $\left(W_b\left(N_b\right)\right)$ denotes the weight of an N_a (N_b)-step movement on G_a (G_b).

Ultimately one is interested in calculating the Gibbs free energy per lattice site in the thermodynamic limit, i.e. from (2.1), (2.2), (2.7) and (2.8)

$$\begin{aligned} g\left(\beta, p_\uparrow, p_\downarrow, \tau\right) &= -\lim_{N\to\infty} \frac{1}{N\beta} \ln Z_G \\ &= \beta^{-1} \ln\left(1 - p_\uparrow - p_\downarrow\right) - \beta^{-1} \sum_{n=1}^{\infty} \frac{\tau^n}{n!} g_n\left(p_\uparrow, p_\downarrow\right) \end{aligned} \tag{2.12}$$

where $g_n\left(p_\uparrow, p_\downarrow\right)$ is equal to the coefficient of N in the expectation value appearing in the sum on the right-hand side of (2.1). From the above remarks and (2.8), $g_n\left(p_\uparrow, p_\downarrow\right)$ is a multinomial of degree n in $p_\uparrow$ and $p_\downarrow$.

In zero field $h = 0$, $z_\uparrow = z_\downarrow = z$, $p_\uparrow = p_\downarrow = p$ and $q = 1 - 2p$, and (2.12) can then be expressed in the form

$$g(\beta, p, \tau) = \beta^{-1} \ln(1-2p) - \beta^{-1} p(1-2p) \sum_{n=2}^{\infty} \frac{\tau^n}{n!} \sum_{i=0}^{n-2} a_i^{(n)} p^i \tag{2.13}$$

where in re-expressing the second term in (2.12), we have made use of the fact that each contributing graph must have at least one particle and one hole.

Using the convention $a_i^{(n)} = 0$ for $i < 0$ and $i > n-2$, the particle density in zero field is given from (2.13) by

$$\rho = -\frac{\partial g}{\partial \mu} = 2p + p(1-2p) \sum_{n=2}^{\infty} \frac{\tau^n}{n!} \sum_{i=0}^{n-1} D_i^{(n)} p^i \tag{2.14}$$

where

$$D_i^{(n)} = (i+1)\left[a_i^{(n)} - 2a_{i-1}^{(n)}\right]. \tag{2.15}$$

Similarly, from (2.12) the zero-field susceptibility can be written as

$$\chi = -\left(\frac{\partial^2 g}{\partial h^2}\right)_{h=0} = -\beta^2 p \left[p\frac{\partial^2 g}{\partial p_\uparrow^2} - 2p\frac{\partial^2 g}{\partial p_\uparrow \partial p_\downarrow} + p\frac{\partial^2 g}{\partial p_\downarrow^2} + (1-2p)\left(\frac{\partial g}{\partial p_\uparrow} + \frac{\partial g}{\partial p_\downarrow}\right)\right]_{p_\uparrow = p_\downarrow = p}$$

$$= \beta\rho + 2\beta p^2(1-2p) \sum_{n=3}^{\infty} \frac{\tau^n}{n!} \sum_{i=0}^{n-3} b_i^{(n)} p^i \tag{2.16}$$

and the zero-field specific heat as

$$C_0 = kp(1-2p) \sum_{n=2}^{\infty} \frac{\tau^n}{n!} \sum_{i=0}^{n-2} c_i^{(n)} p^i \tag{2.17}$$

where k is Boltzmann's constant.

The series (2.14) can also be 'reverted' to re-express the coefficients of τ^n in (2.13), (2.16) and (2.17) as polynomials in the density ρ so that we can write

$$g(\beta, \rho) = \beta^{-1} \ln(1-2p) - (2\beta)^{-1} \rho(1-\rho) \sum_{n=2}^{\infty} \frac{\tau^n}{n!} \sum_{i=0}^{n-2} A_i^{(n)} (\rho/2)^i \tag{2.18}$$

$$\chi(\beta, \rho) = \beta\rho + \frac{\beta}{2}\rho^2(1-\rho) \sum_{n=3}^{\infty} \frac{\tau^n}{n!} \sum_{i=0}^{n-3} B_i^{(n)} (\rho/2)^i \tag{2.19}$$

and

$$C_0(\beta, \rho) = \frac{k\rho}{2}(1-\rho) \sum_{n=2}^{\infty} \frac{\tau^n}{n!} \sum_{i=0}^{n-2} C_i^{(n)} (\rho/2)^i . \tag{2.20}$$

We note in passing that after transforming to operators

$$c_{l,\sigma} = \frac{1}{\sqrt{2}}\left(a_{l,\sigma} + i a_{l,-\sigma}\right) \qquad c_{l,\sigma}^\dagger = \frac{1}{\sqrt{2}}\left(a_{l,\sigma}^\dagger - i a_{l,-\sigma}^\dagger\right) \tag{2.21}$$

the susceptibility series can also be generated from the zero-field fluctuation relation

$$\chi = 2\beta\rho + \frac{2\beta}{N}\sum_{n=2}^{\infty}\frac{\tau^n}{n!}\sum_{i\neq j}\langle S_i^+ S_j^- \mathcal{H}_0^n\rangle_{0,N} \tag{2.22}$$

where the spin operators S_i^+ and S_j^- denote respectively $c_{i\uparrow}^\dagger c_{i\downarrow}$ and $c_{j\downarrow}^\dagger c_{j\uparrow}$ and the subscript N means that we only retain the term proportional to N in the unperturbed expectation value. In this formulation, it is easily proved that the only non-vanishing terms arise from particle movements in which at least two of the particles exchange their positions. In other words, only the particle movements from two or more particles contribute to χ. The general requirement that there is at least one hole leads to the form of (2.16).

3. The infinite-dimensional limit

In order to study the high-density limit we need to scale the hopping energy by $d^{-1/2}$ before taking the limit of lattice dimensionality d to infinity. Thus if we consider for simplicity d-dimensional hypercubic lattices, only terms with n even contribute in (2.1), and we obtain

$$Z_{\mathrm{G}} = Z_0\left\{1+\sum_{n=1}^{\infty}\frac{\tau^{2n}}{d^n(2n)!}\left\langle\left[\sum_{(i,j),\sigma}\left(\tilde{a}_{i,\sigma}^\dagger\tilde{a}_{j\sigma}+\tilde{a}_{j\sigma}^\dagger\tilde{a}_{i\sigma}\right)\right]^{2n}\right\rangle_0\right\}. \tag{3.1}$$

Since movements of particles which contribute to (3.1) must return configurations to their initial state, it is clear that $2n$-step movements can extend into at most n dimensions. If such movements extend into m of the available $d > m$ dimensions, it follows that the total number of these movements is $d!/(d-m)!m!$ times the number of allowed $2n$-step movements in an m-dimensional subspace. These movements then contribute a factor of order $d^{m-n}/m!$ for large d and it follows that the only $2n$-step movements which contribute in the limit $d \to \infty$ are those which extend fully into n dimensions.

By analogy with (2.12) the Gibbs free energy in the infinite-dimensional limit can thus be expressed in the form

$$g\left(\beta,p_\uparrow,p_\downarrow\right) = \beta^{-1}\ln\left(1-p_\uparrow-p_\downarrow\right) - \beta^{-1}\sum_{n=1}^{\infty}\frac{\tau^n}{(2n)!n!}g_{2n}\left(p_\uparrow,p_\downarrow\right) \tag{3.2}$$

where $g_{2n}\left(p_\uparrow,p_\downarrow\right)$ is now the coefficient of N of terms in the expectation value appearing in the sum over n in (3.1) which arise from particle movements of $2n$ steps on bare graphs which extend fully into n dimensions. The expansions of g, χ, C_V and ρ

in parameter p in the infinite-dimensional limit are the following:

$$\begin{aligned}
g &= \frac{1}{\beta}\ln(1-2p) - \frac{1}{\beta}p(1-2p)\sum_{n=1}^{\infty}\frac{\tau^{2n}}{(2n)!n!}\sum_{i=0}^{2n-2}a_i^{(2n)}p^i \\
\chi &= \beta\rho + 2\beta p^2(1-2p)\sum_{n=2}^{\infty}\frac{\tau^{2n}}{(2n)!n!}\sum_{i=0}^{2n-3}b_i^{(2n)}p^i \\
C_V &= kp(1-2p)\sum_{n=1}^{\infty}\frac{\tau^{2n}}{(2n)!n!}\sum_{i=0}^{2n-2}c_i^{(2n)}p^i \\
\rho &= 2p + p(1-2p)\sum_{n=1}^{\infty}\frac{\tau^{2n}}{(2n)!n!}\sum_{i=0}^{2n-1}D_i^{(2n)}p^i .
\end{aligned} \tag{3.3}$$

Equation (3.3) can also be expanded in electron density ρ as the following:

$$\begin{aligned}
g &= \frac{1}{\beta}\ln(1-2p) - \frac{1}{2\beta}\rho(1-\rho)\sum_{n=1}^{\infty}\frac{\tau^{2n}}{(2n)!n!}\sum_{i=0}^{2n-2}A_i^{(2n)}(\rho/2)^i \\
\chi &= \beta\rho + \frac{\beta}{2}\rho^2(1-\rho)\sum_{n=2}^{\infty}\frac{\tau^{2n}}{(2n)!n!}\sum_{i=0}^{2n-3}B_i^{(2n)}(\rho/2)^i \\
C_V &= \frac{k}{2}\rho(1-\rho)\sum_{n=1}^{\infty}\frac{\tau^{2n}}{(2n)!n!}\sum_{i=0}^{2n-2}C_i^{(2n)}(\rho/2)^i .
\end{aligned} \tag{3.4}$$

The expansion coefficients in (3.3) and (3.4) are tabulated in table 1 and table 2.

So while the complexity of the problem is considerably reduced we still need to calculate

(i) the number of embeddings of maximally extended bare graphs;

(ii) the number of possible particle movements on such graphs;

(iii) the sign and magnitude of the weights corresponding to the allowed movements.

Connected and disconnected graphs contribute but in the latter case we only require the appropriate lattice constants, i.e. the coefficients of N in the number of embeddings. These are given up to order ten in tables 3 and 4 of the appendix.

Fortunately, the requirement that $2n$-step movements must extend into n dimensions, and return the configuration to its initial state, drastically reduces the number of allowed graphs. Thus the relevant graphs for $n = 4$, for example, can be conveniently classified into 10 classes and for $n = 5$ into 27 classes. Moreover, if the edges of a graph which are part of a loop are labelled '1-edges' and the other edges are labelled as '2-edges', the label of an edge is equal to the number of times a particle must traverse that edge in an allowed movement on the graph. It follows that by keeping track of movements on edges, we can discard those movements for which any given edge label is exceeded. This, of course, reduces the enumeration problem considerably. The aforementioned considerations for calculating the weights associated with an allowed movement apply here also and the results are given in table 5 of the appendix.

Table 1. The high-T expansion coefficients in terms of p defined in (3.3) for the strongly correlated infinite-dimensional Hubbard model on the hypercubic lattice to order τ^{10}.

	n	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$	$i=9$
$a_i^{(n)}$	2	4									
	4	48	-512	976							
	6	1440	-68736	610176	-1815936	1728384					
	8	80640	-14337024	337378560	-2760778752	10087834368	-16969393152	10730409216			
	10	7257600	-4689807360	250917396480	-4254101268480	33105894044160	-135982335037440	305596106833920	-355795430430720	168041954365440	
$b_i^{(n)}$	4	-64	176								
	6	-17664	253824	-938880	1014528						
	8	-5984256	245164800	-2588703744	10987292160	-20415135744	13862116608				
	10	-2712944640	275984363520	-6291117158400	58237599851520	-268396952524800	654923954196480	-811221317498880	402121537536000		
$c_i^{(n)}$	2	8									
	4	192	-3072	5568							
	6	8640	-725760	7361280	-21945600	20148480					
	8	645120	-244457472	6875301888	-59513954304	218334627840	-360342872064	221232420864			
	10	72576000	-122356224000	8087239065600	-148068452044800	1181998851609600	-4853975904460800	10757770699161600	-12260265540710400	5645529053798400	
$D_i^{(n)}$	2	4	-16								
	4	48	-1216	6000	-7808						
	6	1440	-143232	2242944	-12145152	26801280	-20740608				
	8	80640	-28996608	1098157824	-13742143488	78046959360	-222870371328	312684368640	-171686547456		
	10	7257600	-9408645120	780891033600	-19023744245760	208070482905600	-1213164738754560	4042925438361600	-7735901152788480	7916695337041920	-3360839087308800

Table 2. The high-T expansion coefficients in terms of ρ defined in (3.4) for the strongly correlated infinite-dimensional Hubbard model on the hypercubic lattice to order τ^{10}.

	n	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	$i=7$	$i=8$
$A_i^{(n)}$	2	4								
	4	-48	256	-560						
	6	1440	-5376	2496	38784	-22656				
	8	-80640	1446912	-29888256	234857472	-874442496	1501166592	-965702400		
	10	7257600	-46325760	-11313976320	251145861120	-2118114608640	9086882703360	-20885920250880	24424745441280	-114033419567360
$B_i^{(n)}$	4	-64	176							
	6	-6144	125664	-483840	507648					
	8	-651264	68283648	-883092480	3926120448	-7220797440	4732722432			
	10	-114032640	45387717120	-1502800128000	15887733227520	-76354988697600	186208161876480	-225442430484480	107885704396800	
$C_i^{(n)}$	2	8								
	4	0	-1536	2496						
	6	0	-161280	2450880	-7303680	6232320				
	8	0	-35782656	1619208192	-16156385280	59856039936	-94964244480	54917388288		
	10	0	-10638950400	1456428211200	-33167464243200	284795185766400	-1177524551577600	2543473072972800	-2777856148684800	1214881271193600

The main results, in table 1, give the coefficients in the high-temperature expansion for the Gibbs free energy, the zero field susceptibility, specific heat, and density as functions of the parameter p defined in (3.3). Coefficients of the corresponding series (3.4) expressed in terms of the density are given in table 2.

4. Numerical analysis

We begin by re-expressing the susceptibility and specific heat series (3.4) in the form

$$\chi(\rho,\tau) = \beta \sum_{n=0}^{\infty} u_n(p)\tau^{2n} = \beta \sum_{n=0}^{\infty} U_n(\rho)\tau^{2n} \tag{4.1}$$

and

$$C(\rho,\tau) = k \sum_{n=1}^{\infty} v_n(p)\tau^{2n} = k \sum_{n=1}^{\infty} V_n(\rho)\tau^{2n} \tag{4.2}$$

where $u_0(p) = 2p$, $U_0(\rho) = \rho$, $U_1(\rho) = 0$ and

$$u_n(p) = \frac{p(1-2p)}{(2n)!n!} \sum_{i=0}^{2n-1} \bar{b}_i^{(2n)} p^i \qquad n \geq 1$$

$$U_n(\rho) = \frac{\rho^2(1-\rho)}{2(2n)!n!} \sum_{i=0}^{2n-3} B_i^{(2n)} (\rho/2)^i \qquad n \geq 2 \tag{4.3}$$

$$v_n(p) = \frac{p(1-2p)}{(2n)!n!} \sum_{i=0}^{2n-2} c_i^{(2n)} p^i$$

$$V_n(\rho) = \frac{\rho(1-\rho)}{2(2n)!n!} \sum_{i=0}^{2n-2} C_i^{(2n)} (\rho/2)^i \qquad n \geq 1 \tag{4.4}$$

$\bar{b}_i^{(2n)}$ is defined as the following

$$\bar{b}_i^{(2n)} = \begin{cases} D_i^{(2n)} & \text{when } i = 0 \text{ and } i = 2n-1 \\ D_i^{(2n)} + 2b_{i-1}^{(2n)} & \text{when } 1 \leq i \leq 2n-2 \end{cases} \tag{4.5}$$

and the coefficients $b_i^{(2n)}$, $c_i^{(2n)}$, $D_i^{(2n)}$, $B_i^{(2n)}$ and $C_i^{(2n)}$ are given in table 1 and table 2.

It will be noted from the table that for a fixed density ρ, the series coefficients $U_n(\rho)$ and $V_n(\rho)$ have irregular signs so that a straightforward ratio test is inappropriate.

For $\rho \geq 0.4$, the susceptibility series alternate in sign, as do the specific heat series for $\rho \geq 0.5$. This sign pattern is consistent with a dominant singularity on the negative real τ^2 axis (i.e. with an imaginary temperature) at $\tau^2 \simeq -0.5$ for $\rho \simeq 0.4$ and $\tau^2 \simeq -0.6$ for $\rho \simeq 0.5$. Analysis of the series by Padé approximants and other differential approximants gave results consistent with this observation. However, the series are still too short to say anything other than that there is no evidence whatever of a singularity on the positive real τ axis for any value of ρ in the range $0.1 \leq \rho \leq 0.9$, and hence no evidence for a phase transition. A similar analysis for fixed p in the range $0.05 \leq p \leq 0.45$ gives consistent results. That is, no evidence of a singularity on the positive real τ axis for any value of p in this range.

An analysis of the two-dimensional series, which are one term shorter, also shows no evidence of a phase transition.

In order to further test our finding that the limiting model shows no evidence of a phase transition, we used the actual series (4.1) and the known coefficients to calculate ρ/χ for various τ^{-1} at particular values of the density. The results given in figure 2, although with apparent even–odd oscillation when τ approaches its limit of convergence $\tau = 1$, are again consistent with the conclusion that the susceptibility only diverges at zero temperature.

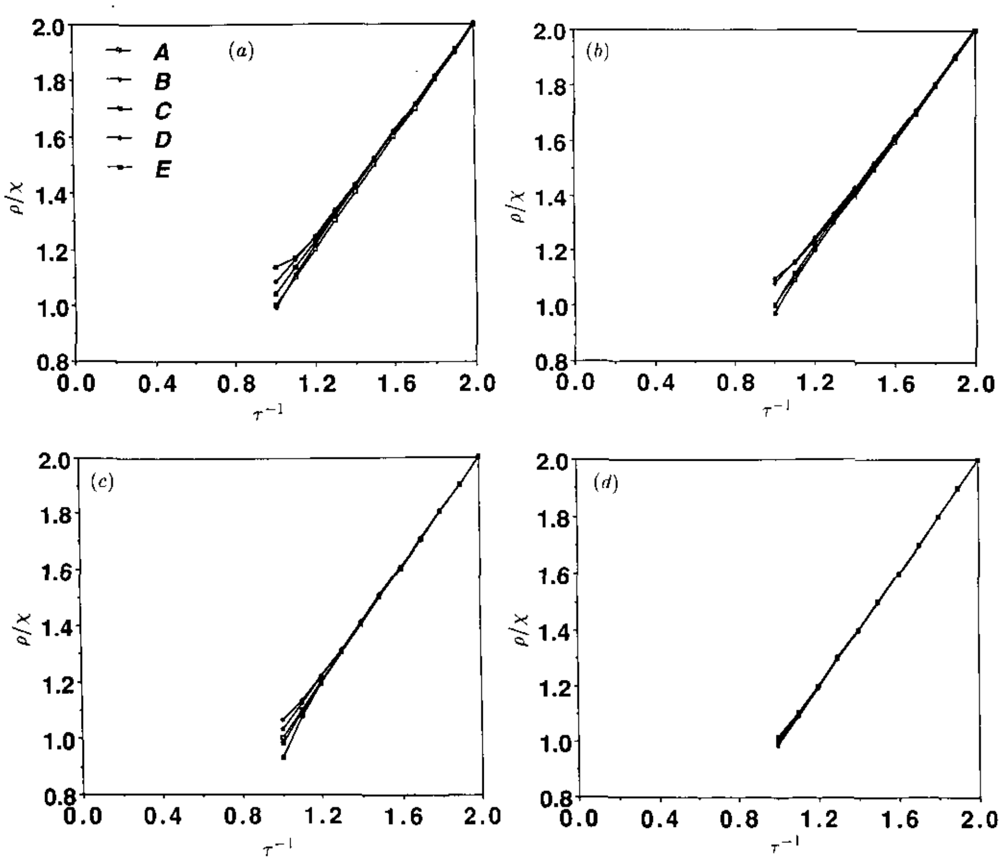

Figure 2. ρ/χ, computed from the series for particular values of the density ρ, versus τ^{-1}. (a)–(d) are the plots for $\rho = 0.2$, 0.4, 0.6 and 0.8. (A)–(E) show the results expanded to order τ^2, τ^4, τ^6, τ^8 and τ^{10}.

Our results are also consistent with the recent rigorous results for the Falicov–Kimball model [26], which is simply a Hubbard model where only one spin species of particle is allowed to hop. In the infinite-dimensional and $U \to \infty$ limits, this model does not, it seems, have a phase transition at a finite critical temperature for any density.

5. Discussion

It is generally believed [11, 27–29] that at low hole concentration the ground state of the strongly correlated Hubbard model is ferromagnetic. There is no common agreement, however, regarding the existence of a ferromagnetic state, or phase transition at a finite temperature. In this paper, we have studied the strongly correlated Hubbard model in the infinite-dimensionality limit using the method of high-temperature series analysis and find no evidence for a magnetic phase transition at any hole density. Since critical temperatures usually increase with increasing dimensionality, we conjecture that the strongly correlated Hubbard model does not undergo a magnetic transition in any dimension.

In order to check our results we have generated high-temperature expansions in which only the dominant maximally extended polygons are taken into account. These series behave in precisely the same way as the shorter exact expansions, as do corresponding exact high-temperature expansions for the related Falicov–Kimball model which was shown recently [26] to have no magnetic transition in the infinite dimensional and $U \to \infty$ limits. These studies will be published elsewhere.

Finally we should point out that our results do not exclude more exotic types of phase transitions of say the Kosterlitz–Thouless type where the magnetic susceptibility is finite at non-zero temperature, but where correlations have a power-law decay below a certain critical temperature.

Acknowledgments

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Appendix

C_a, C_b, C_c, $C_d + C_e$, C_f, C_g, $C_h + C_i + C_j$, $C_k + C_l + C_m + C_n$, C_o, $C_p + C_q$ in table 3 were confirmed by exact enumeration.

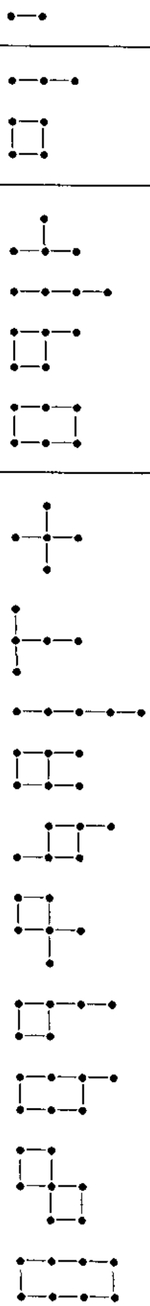
Defining $K_d(n)$ as the number of n-bond lattice animals on the d-dimensional hypercubic lattice, and $S_d(n)$ as the number of n-bond lattice animals which extend into d-dimensions on the d-dimensional hypercubic lattice, the following relation exists:

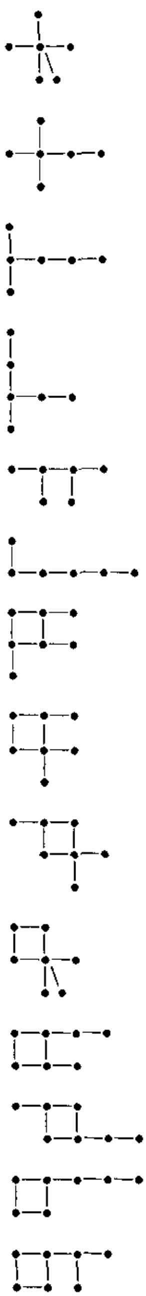
$$K_d(n) = \sum_{d'=1}^{d} \frac{d!}{(d-d')!d'!} S_{d'}(n). \tag{A1}$$

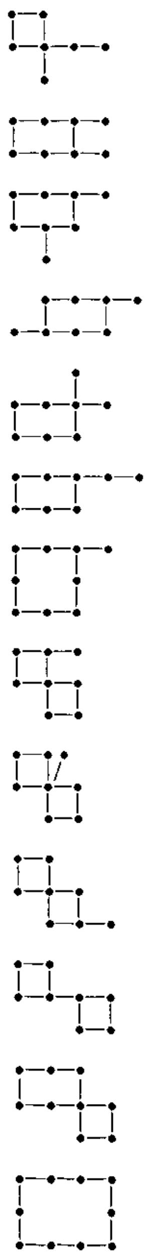
From the known data for $K_d(n)$ [30], we can solve $S_d(n)$ from (A1). It is readily seen that

$$\begin{aligned} C_b &= S_2(2) = 4 \\ C_d + C_e &= S_3(3) = 32 \\ C_h + C_i + C_j &= S_4(4) = 400 \\ C_r + C_s + C_t + C_u + C_v + C_w &= S_5(5) = 6912. \end{aligned} \tag{A2}$$

Table 3. The number of maximally extended connected graphs per lattice site C_α of class α for movements up to 10 steps on hypercubic lattices. C_α for $\alpha = a$; $\alpha = b$ and c; $\alpha = d, e, f, g$; $\alpha = h, i, \ldots, q$ and $\alpha = r, s, \ldots, R$ are counted on one-, two-, three-, four- and five-dimensional hypercubic lattices respectively.

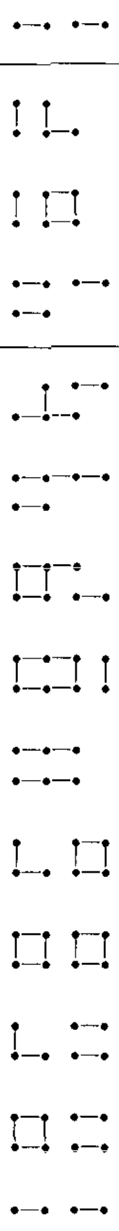




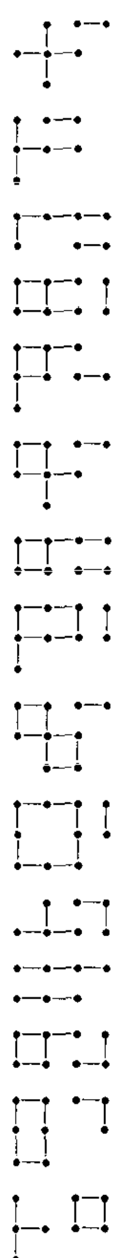


α	Typical Graph	C_α
a		1
b		4
c		1
d		8
e		24
f		24
g		16
h		16
i		192
j		192
k		192
l		96
m		96
n		192
o		768
p		48
q		648
r		32
s		640
t		1920
u		1920
v		480
w		1920
x		1920
y		1920
z		960
A		320
B		3840
C		1920
D		1920
E		960
F		1920
G		7680
H		7680
I		3840
J		3840
K		7680
L		51840
M		1920
N		480
O		960
P		480
Q		3840
R		47616

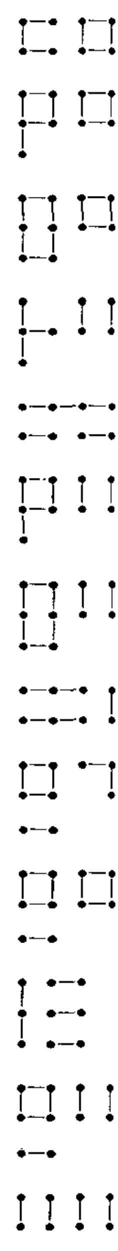
Table 4. The N coefficient of the number of maximally extended disconnected graphs per lattice site C_α of class α on an N-site hypercubic lattice lattice for movements up to ten steps. C_α for $\alpha = (a^2)$; $\alpha = (a,b),(a,c)$ and (a^3); $\alpha = (a,d),(a,e),\ldots,(a^4)$ and $\alpha = (a,h),(a,i),\ldots,(a^5)$ are counted on two-, three-, four- and five-dimensional hypercubic lattices respectively.



α	Typical Graph	C_α
(a^2)		−4
(a,b)		−72
(a,c)		−24
(a^3)		40
(a,d)		−256
(a,e)		−768
(a,f)		−960
(a,g)		−768
(b^2)		−432
(b,c)		−288
(c^2)		−48
(a^2,b)		1728
(a^2,c)		672
(a^4)		−672



α	Typical Graph	C_α
(a,h)		−800
(a,i)		−9600
(a,j)		−9600
(a,k)		−11520
(a,l)		−5760
(a,m)		−5760
(a,n)		−11520
(a,o)		−53760
(a,p)		−3360
(a,q)		−51840
(b,d)		−3840
(b,e)		−11520
(b,f)		−14400
(b,g)		−11520
(c,d)		−1280



α	Typical Graph	C_α
(c,e)		−3840
(c,f)		−4800
(c,g)		−3840
(a^2,d)		8960
(a^2,e)		26880
(a^2,f)		38400
(a^2,g)		34560
(a,b^2)		30240
(a,b,c)		23040
(a,c^2)		4320
(a^3,b)		−53760
(a^3,c)		−23040
(a^5)		16128

Table 5. The zero field weight expansion coefficients of each connected graph defined in (A4) for high-T expansion of the strongly correlated infinite dimensional Hubbard model to order τ^{10}.

N	α	n_α	$K_m^{(\alpha)}$								
			$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$
2	a	2	4								
4	b	3	8	16							
	c	4	16	-64	16						
6	d	4	24	120	96						
	e	4	12	288	48						
	f	5	24	-24	-480	48					
	g	6	24	-624	1584	-624	24				
8	h	5	96	960	1632	768					
	i	5	32	2176	4352	256					
	j	5	16	3232	6464	128					
	k	6	32	-256	-2720	-6784	128				
	l	6	32	-256	-2720	-6784	128				
	m	6	64	768	-3776	-4224	256				
	n	6	32	4672	-18848	2176	128				
	o	7	32	-2048	-3136	20224	-5536	64			
	p	7	64	5760	-17920	27904	-2112	128			
	q	8	32	-3840	38112	-77312	38112	-3840	32		
10	r	6	480	9120	24480	23520	7680				
	s	6	120	17280	80040	64800	1920				
	t	6	40	32120	148320	128480	640				
	u	6	40	16360	229280	65440	640				
	v	6	80	25120	102720	100480	1280				
	w	6	20	25840	304800	103360	320				
	x	7	40	-8200	-71360	-102160	-105920	320			
	y	7	80	400	-19120	-119120	-99040	640			
	z	7	80	400	-19120	-119120	-99040	640			
	A	7	240	10800	-27600	-84720	-44640	1920			
	B	7	40	19760	-27080	-440560	-10400	320			
	C	7	40	19760	-27080	-440560	-10400	320			
	D	7	40	47720	-74000	-598480	85120	320			
	E	7	80	56320	-107200	-423440	92000	640			
	F	7	80	24800	-17280	-365040	28960	640			
	G	8	40	-12160	-67360	41120	437240	-62080	160		
	H	8	40	-12160	-67360	41120	437240	-62080	160		
	I	8	40	-12160	-67360	41120	437240	-62080	160		
	J	8	80	-7520	-120560	197200	242080	-67760	320		
	K	8	40	15800	-559480	1359320	-378160	-14320	160		
	L	9	40	-16120	151200	434240	-1499960	616680	-35600	80	
	M	8	80	31600	80560	-164240	564160	-28640	320		
	N	8	160	33760	-128000	118400	271840	-7520	640		
	O	8	80	31600	80560	-164240	564160	-28640	320		
	P	8	80	87520	-1086080	2470240	-886160	66880	320		
	Q	9	80	23680	-555120	1717600	-2086480	432000	-23440	160	
	R	10	40	-20080	584320	-3529360	6247600	-3529360	584320	-20080	40

It is easily seen that the following sum rules should be satisfied:

$$
\begin{aligned}
&C_b + C_{(a^2)} = N^2 \\
&C_d + C_e + C_{(a,b)} + C_{(a^3)} = N^3 \\
&C_f + C_{(a,c)} = C_2^3 N^2 \\
&C_h + C_i + C_j + C_{(a,d)} + C_{(a,e)} + C_{(b^2)} + C_{(a^2,b)} + C_{(a^4)} = N^4 \\
&C_k + C_l + C_m + C_n + C_{(af)} + C_{(b,c)} + C_{(a^2,c)} = C_2^4 N^3 \\
&C_o + C_{(a,g)} = C_3^4 16 N^2 \\
&C_p + C_{(c^2)} = C_2^4 N^2/2! \\
&C_r + C_s + C_t + C_u + C_v + C_w + C_{(a,h)} + C_{(a,i)} + C_{(a,j)} + C_{(b,d)} \\
&\quad + C_{(b,e)} + C_{(a^2,d)} + C_{(a^2,e)} + C_{(a,b^2)} + C_{(a^3,b)} + C_{a^5)} = N^5 \\
&C_x + C_y + C_z + C_A + C_B + C_C + C_D + C_E + C_F + C_{(a,k)} + C_{(a,l)} + C_{(a,m)} \\
&\quad + C_{(a,n)} + C_{(b,f)} + C_{(c,d)} + C_{(c,e)} + C_{(a^2,f)} + C_{(a,b,c)} + C_{(a^3,c)} = C_2^5 N^4 \\
&C_G + C_H + C_I + C_J + C_K + C_{(a,o)} + C_{(b,g)} + C_{(a^2,g)} = C_3^5 16 N^3 \\
&C_L + C_{(a,q)} = C_4^5 648 N^2 \\
&C_M + C_N + C_O + C_P + C_{(a,p)} + C_{(c,f)} + C_{(a,c^2)} = C_2^5 C_2^3 N^3 \\
&C_Q + C_{(c,g)} = C_3^5 16 C_2^2 N^2.
\end{aligned}
\tag{A3}
$$

Here $C_m^n \equiv n!/(n-m)!m!$. (A2) and (A3) are consistent with the data in table 3 and table 4.

The zero field weight W_α for each connected graph in class α is given by the following:

$$
W_\alpha = \sum_{m=1}^{n_\alpha - 1} K_m^{(\alpha)} p^m q^{n_\alpha - m}. \tag{A4}
$$

Where the values of n_α and $K_m^{(\alpha)}$ for expansions to order of τ^{10} are tabulated in table 5.

The finite field Gibbs free energy per lattice site for the Hubbard model in the strong correlation and infinite-dimensional limit is the following:

$$
\begin{aligned}
g(\beta, p_\uparrow, p_\downarrow, t) &= \frac{1}{\beta}\ln(1 - p_\uparrow - p_\downarrow) - \frac{1}{\beta}\frac{\tau^2}{2!1!}\left[2p_\uparrow q + 2p_\downarrow q\right] \\
&- \frac{1}{\beta}\frac{\tau^4}{4!2!}\ \left[8p_\uparrow^3 q + 16p_\uparrow^2 q + 32p_\uparrow p_\downarrow q + 16p_\downarrow^2 q + 8p_\downarrow^3 q - 128p_\uparrow^2 q^2 + 16p_\uparrow q^2 - 192p_\uparrow p_\downarrow q^2 + 16p_\downarrow q^2 \right. \\
&\quad \left. - 128p_\downarrow^2 q^2 + 8p_\uparrow q^3 + 8p_\downarrow q^3\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\beta}\frac{\tau^6}{6!3!}\Big[192p_\uparrow^5q + 288p_\uparrow^4q + 240p_\uparrow^3q + 288p_\uparrow^3p_\downarrow q + 720p_\uparrow^2p_\downarrow q + 720p_\uparrow p_\downarrow^2q + 288p_\uparrow p_\downarrow^3q + 240p_\downarrow^3q \\
&\quad + 288p_\downarrow^4q + 192p_\downarrow^5q - 10752p_\uparrow^4q^2 - 10656p_\uparrow^3q^2 - 5760p_\uparrow^3p_\downarrow q^2 + 1920p_\uparrow^2q^2 - 29664p_\uparrow^2p_\downarrow q^2 \\
&\quad + 4032p_\uparrow p_\downarrow q^2 - 29664p_\uparrow p_\downarrow^2q^2 - 5760p_\uparrow p_\downarrow^3q^2 + 1920p_\downarrow^2q^2 - 10656p_\downarrow^3q^2 - 10752p_\downarrow^4q^2 \\
&\quad + 64512p_\uparrow^3q^3 - 10656p_\uparrow^2q^3 + 109440p_\uparrow^2p_\downarrow q^3 + 240p_\uparrow q^3 - 13824p_\uparrow p_\downarrow q^3 + 109440p_\uparrow p_\downarrow^2q^3 \\
&\quad + 240p_\downarrow q^3 - 10656p_\downarrow^2q^3 + 64512p_\downarrow^3q^3 - 10752p_\uparrow^2q^4 + 288p_\uparrow q^4 - 11520p_\uparrow p_\downarrow q^4 + 288p_\downarrow q^4 \\
&\quad - 10752p_\downarrow^2q^4 + 192p_\uparrow q^5 + 192p_\downarrow q^5\Big] \\
&-\frac{1}{\beta}\frac{\tau^8}{8!4!}\Big[10368p_\uparrow^7q + 13824p_\uparrow^6q + 10752p_\uparrow^5q + 12288p_\uparrow^5p_\downarrow q + 5376p_\uparrow^4q + 21504p_\uparrow^4p_\downarrow q \\
&\quad + 21504p_\uparrow^3p_\downarrow q + 10752p_\uparrow^3p_\downarrow^2q + 3072p_\uparrow^3p_\downarrow^3q + 32256p_\uparrow^2p_\downarrow^2q + 10752p_\uparrow^2p_\downarrow^3q + 21504p_\uparrow p_\downarrow^3q \\
&\quad + 21504p_\uparrow p_\downarrow^4q + 12288p_\uparrow p_\downarrow^5q + 5376p_\downarrow^4q + 10752p_\downarrow^5q + 13824p_\downarrow^6q + 10368p_\downarrow^7q \\
&\quad - 1975296p_\uparrow^6q^2 - 2231808p_\uparrow^5q^2 - 516096p_\uparrow^5p_\downarrow q^2 - 935424p_\uparrow^4q^2 - 3735552p_\uparrow^4p_\downarrow q^2 \\
&\quad + 252672p_\uparrow^3q^2 - 4131840p_\uparrow^3p_\downarrow q^2 - 1370112p_\uparrow^3p_\downarrow^2q^2 - 430080p_\uparrow^3p_\downarrow^3q^2 + 798720p_\uparrow^2p_\downarrow q^2 \\
&\quad - 6429696p_\uparrow^2p_\downarrow^2q^2 - 1370112p_\uparrow^2p_\downarrow^3q^2 + 798720p_\uparrow p_\downarrow^2q^2 - 4131840p_\uparrow p_\downarrow^3q^2 - 3735552p_\uparrow p_\downarrow^4q^2 \\
&\quad - 516096p_\uparrow p_\downarrow^5q^2 + 252672p_\downarrow^3q^2 - 935424p_\downarrow^4q^2 - 2231808p_\downarrow^5q^2 - 1975296p_\downarrow^6q^2 \\
&\quad + 36518784p_\uparrow^5q^3 + 20330496p_\uparrow^4q^3 + 31481856p_\uparrow^4p_\downarrow q^3 - 5763072p_\uparrow^3q^3 + 69491712p_\uparrow^3p_\downarrow q^3 \\
&\quad + 10752000p_\uparrow^3p_\downarrow^2q^3 + 252672p_\uparrow^2q^3 - 14596608p_\uparrow^2p_\downarrow q^3 + 92694528p_\uparrow^2p_\downarrow^2q^3 + 10752000p_\uparrow^2p_\downarrow^3q^3 \\
&\quad + 548352p_\uparrow p_\downarrow q^3 - 14596608p_\uparrow p_\downarrow^2q^3 + 69491712p_\uparrow p_\downarrow^3q^3 + 31481856p_\uparrow p_\downarrow^4q^3 + 252672p_\downarrow^2q^3 \\
&\quad - 5763072p_\downarrow^3q^3 + 20330496p_\downarrow^4q^3 + 36518784p_\downarrow^5q^3 - 126203904p_\uparrow^4q^4 + 20330496p_\uparrow^3q^4 \\
&\quad - 215126016p_\uparrow^3p_\downarrow q^4 - 935424p_\uparrow^2q^4 + 28354560p_\uparrow^2p_\downarrow q^4 - 241704960p_\uparrow^2p_\downarrow^2q^4 + 5376p_\uparrow q^4 \\
&\quad - 887808p_\uparrow p_\downarrow q^4 + 28354560p_\uparrow p_\downarrow^2q^4 - 215126016p_\uparrow p_\downarrow^3q^4 + 5376p_\downarrow q^4 - 935424p_\downarrow^2q^4 \\
&\quad + 20330496p_\downarrow^3q^4 - 126203904p_\downarrow^4q^4 + 36518784p_\uparrow^3q^5 - 2231808p_\uparrow^2q^5 + 42233856p_\uparrow^2p_\downarrow q^5 \\
&\quad + 10752p_\uparrow q^5 - 1993728p_\uparrow p_\downarrow q^5 + 42233856p_\uparrow p_\downarrow^2q^5 + 10752p_\downarrow q^5 - 2231808p_\downarrow^2q^5 \\
&\quad + 36518784p_\downarrow^3q^5 - 1975296p_\uparrow^2q^6 + 13824p_\uparrow q^6 - 1462272p_\uparrow p_\downarrow q^6 + 13824p_\downarrow q^6 \\
&\quad - 1975296p_\downarrow^2q^6 + 10368p_\uparrow q^7 + 10368p_\downarrow q^7\Big] \\
&-\frac{1}{\beta}\frac{\tau^{10}}{10!5!}\Big[952320p_\uparrow^9q + 1190400p_\uparrow^8q + 864000p_\uparrow^7q + 1036800p_\uparrow^7p_\downarrow q + 460800p_\uparrow^6q \\
&\quad + 1555200p_\uparrow^6p_\downarrow q + 161280p_\uparrow^5q + 1382400p_\uparrow^5p_\downarrow q + 691200p_\uparrow^5p_\downarrow^2q + 153600p_\uparrow^5p_\downarrow^3q \\
&\quad + 806400p_\uparrow^4p_\downarrow q + 1382400p_\uparrow^4p_\downarrow^2q + 345600p_\uparrow^4p_\downarrow^3q + 1612800p_\uparrow^3p_\downarrow^2q + 921600p_\uparrow^3p_\downarrow^3q \\
&\quad + 345600p_\uparrow^3p_\downarrow^4q + 153600p_\uparrow^3p_\downarrow^5q + 1612800p_\uparrow^2p_\downarrow^3q + 1382400p_\uparrow^2p_\downarrow^4q + 691200p_\uparrow^2p_\downarrow^5q \\
&\quad + 806400p_\uparrow p_\downarrow^4q + 1382400p_\uparrow p_\downarrow^5q + 1555200p_\uparrow p_\downarrow^6q + 1036800p_\uparrow p_\downarrow^7q + 161280p_\downarrow^5q \\
&\quad + 460800p_\downarrow^6q + 864000p_\downarrow^7q + 1190400p_\downarrow^8q + 952320p_\downarrow^9q - 630128640p_\uparrow^8q^2 \\
&\quad - 746688000p_\uparrow^7q^2 - 74649600p_\uparrow^7p_\downarrow q^2 - 403200000p_\uparrow^6q^2 - 969408000p_\uparrow^6p_\downarrow q^2 \\
&\quad - 85939200p_\uparrow^5q^2 - 1187020800p_\uparrow^5p_\downarrow q^2 - 200140800p_\uparrow^5p_\downarrow^2q^2 - 77414400p_\uparrow^5p_\downarrow^3q^2 \\
&\quad + 39352320p_\uparrow^4q^2 - 597657600p_\uparrow^4p_\downarrow q^2 - 1073894400p_\uparrow^4p_\downarrow^2q^2 - 251443200p_\uparrow^4p_\downarrow^3q^2 \\
&\quad + 165312000p_\uparrow^3p_\downarrow q^2 - 1379481600p_\uparrow^3p_\downarrow^2q^2 - 571468800p_\uparrow^3p_\downarrow^3q^2 - 251443200p_\uparrow^3p_\downarrow^4q^2 \\
&\quad - 77414400p_\uparrow^3p_\downarrow^5q^2 + 251904000p_\uparrow^2p_\downarrow^2q^2 - 1379481600p_\uparrow^2p_\downarrow^3q^2 - 1073894400p_\uparrow^2p_\downarrow^4q^2 \\
&\quad - 200140800p_\uparrow^2p_\downarrow^5q^2 + 165312000p_\uparrow p_\downarrow^3q^2 - 597657600p_\uparrow p_\downarrow^4q^2 - 1187020800p_\uparrow p_\downarrow^5q^2 \\
&\quad - 969408000p_\uparrow p_\downarrow^6q^2 - 74649600p_\uparrow p_\downarrow^7q^2 + 39352320p_\downarrow^4q^2 - 85939200p_\downarrow^5q^2 \\
&\quad - 403200000p_\downarrow^6q^2 - 746688000p_\downarrow^7q^2 - 630128640p_\downarrow^8q^2 + 29023887360p_\uparrow^7q^3 \\
&\quad + 23422348800p_\uparrow^6q^3 + 14880153600p_\uparrow^6p_\downarrow q^3 + 4591929600p_\uparrow^5q^3 + 51032140800p_\uparrow^5p_\downarrow q^3
\end{aligned}
$$

$$
\begin{aligned}
&+2399846400p_\uparrow^5p_\downarrow^2q^3-2543846400p_\uparrow^4q^3+29611699200p_\uparrow^4p_\downarrow q^3+34007347200p_\uparrow^4p_\downarrow^2q^3\\
&+5496422400p_\uparrow^4p_\downarrow^3q^3+165473280p_\uparrow^3q^3-9628915200p_\uparrow^3p_\downarrow q^3+61582656000p_\uparrow^3p_\downarrow^2q^3\\
&+17555558400p_\uparrow^3p_\downarrow^3q^3+5496422400p_\uparrow^3p_\downarrow^4q^3+540288000p_\uparrow^2p_\downarrow q^3-14184192000p_\uparrow^2p_\downarrow^2q^3\\
&+61582656000p_\uparrow^2p_\downarrow^3q^3+34007347200p_\uparrow^2p_\downarrow^4q^3+2399846400p_\uparrow^2p_\downarrow^5q^3+540288000p_\uparrow p_\downarrow^2q^3\\
&-9628915200p_\uparrow p_\downarrow^3q^3+29611699200p_\uparrow p_\downarrow^4q^3+51032140800p_\uparrow p_\downarrow^5q^3+14880153600p_\uparrow p_\downarrow^6q^3\\
&+165473280p_\downarrow^3q^3-2543846400p_\downarrow^4q^3+4591929600p_\downarrow^5q^3+23422348800p_\downarrow^6q^3\\
&+29023887360p_\downarrow^7q^3-282321223680p_\uparrow^6q^4-93206284800p_\uparrow^5q^4-295217049600p_\uparrow^5p_\downarrow q^4\\
&+32635699200p_\uparrow^4q^4-370150579200p_\uparrow^4p_\downarrow q^4-159938150400p_\uparrow^4p_\downarrow^2q^4-2543846400p_\uparrow^3q^4\\
&+92649984000p_\uparrow^3p_\downarrow q^4-562187827200p_\uparrow^3p_\downarrow^2q^4-93826252800p_\uparrow^3p_\downarrow^3q^4+39352320p_\uparrow^2q^4\\
&-6076377600p_\uparrow^2p_\downarrow q^4+117339648000p_\uparrow^2p_\downarrow^2q^4-562187827200p_\uparrow^2p_\downarrow^3q^4-159938150400p_\uparrow^2p_\downarrow^4q^4\\
&+87398400p_\uparrow p_\downarrow q^4-6076377600p_\uparrow p_\downarrow^2q^4+92649984000p_\uparrow p_\downarrow^3q^4-370150579200p_\uparrow p_\downarrow^4q^4\\
&-295217049600p_\uparrow p_\downarrow^5q^4+39352320p_\downarrow^2q^4-2543846400p_\downarrow^3q^4+32635699200p_\downarrow^4q^4\\
&-93206284800p_\downarrow^5q^4-282321223680p_\downarrow^6q^4+692866560000p_\uparrow^5q^5-93206284800p_\uparrow^4q^5\\
&+1120164249600p_\uparrow^4p_\downarrow q^5+4591929600p_\uparrow^3q^5-116132390400p_\uparrow^3p_\downarrow q^5+1250784460800p_\uparrow^3p_\downarrow^2q^5\\
&-85939200p_\uparrow^2q^5+3083808000p_\uparrow^2p_\downarrow q^5-124850380800p_\uparrow^2p_\downarrow^2q^5+1250784460800p_\uparrow^2p_\downarrow^3q^5\\
&+161280p_\uparrow q^5-15974400p_\uparrow p_\downarrow q^5+3083808000p_\uparrow p_\downarrow^2q^5-116132390400p_\uparrow p_\downarrow^3q^5\\
&+1120164249600p_\uparrow p_\downarrow^4q^5+161280p_\downarrow q^5-85939200p_\downarrow^2q^5+4591929600p_\downarrow^3q^5\\
&-93206284800p_\downarrow^4q^5+692866560000p_\downarrow^5q^5-282321223680p_\uparrow^4q^6+23422348800p_\uparrow^3q^6\\
&-342130176000p_\uparrow^3p_\downarrow q^6-403200000p_\uparrow^2q^6+22008844800p_\uparrow^2p_\downarrow q^6-319876300800p_\uparrow^2p_\downarrow^2q^6\\
&+460800p_\uparrow q^6-256281600p_\uparrow p_\downarrow q^6+22008844800p_\uparrow p_\downarrow^2q^6-342130176000p_\uparrow p_\downarrow^3q^6\\
&+460800p_\downarrow q^6-403200000p_\downarrow^2q^6+23422348800p_\downarrow^3q^6-282321223680p_\downarrow^4q^6\\
&+29023887360p_\uparrow^3q^7-746688000p_\uparrow^2q^7+22776422400p_\uparrow^2p_\downarrow q^7+864000p_\uparrow q^7\\
&-451276800p_\uparrow p_\downarrow q^7+22776422400p_\uparrow p_\downarrow^2q^7+864000p_\downarrow q^7-746688000p_\downarrow^2q^7\\
&+29023887360p_\downarrow^3q^7-630128640p_\uparrow^2q^8+1190400p_\uparrow q^8-304128000p_\uparrow p_\downarrow q^8\\
&+1190400p_\downarrow q^8-630128640p_\downarrow^2q^8+952320p_\uparrow q^9+952320p_\downarrow q^9\Big]+\mathrm{O}(\tau^{12}).
\end{aligned}
\tag{A5}
$$

The parameters β, $p_\uparrow$, $p_\downarrow$, p, q, t, τ are defined in the text.

When the external magnetic field $h=0$, $p_\uparrow=p$ and $p_\downarrow=p$ and (A5) becomes

$$
\begin{aligned}
g(\beta,p,t)=&\frac{1}{\beta}\ln(1-2p)-\frac{4}{\beta}\frac{\tau^2}{2!1!}pq-\frac{1}{\beta}\frac{\tau^4}{4!2!}pq\big(64p+16p^2+32q-448pq+16q^2\big)\\
&-\frac{1}{\beta}\frac{\tau^6}{6!3!}\,pq\big(1920p^2+1152p^3+384p^4+7872pq-80640p^2q-33024p^3q+480q^2-35136pq^2\\
&\qquad+347904p^2q^2+576q^3-33024pq^3+384q^4\big)\\
&-\frac{1}{\beta}\frac{\tau^8}{8!4!}\,pq\big(86016p^3+86016p^4+55296p^5+20736p^6+2102784p^2q-16564224p^3q-14674944p^4q\\
&\qquad-5412864p^5q+1053696pq^2-40719360p^2q^2+272338944p^3q^2+157505280p^4q^2+10752q^3\\
&\qquad-2758656pq^3+97370112p^2q^3-924364800p^3q^3+21504q^4-6457344pq^4+157505280p^2q^4\\
&\qquad+27648q^5-5412864pq^5+20736q^6\big)\\
&-\frac{1}{\beta}\frac{\tau^{10}}{10!5!}pq\big(5160960p^4+7372800p^5+6912000p^6+4761600p^7+1904640p^8+661232640p^3q\\
&\qquad-4126156800p^4q-5899699200p^5q-4335360000p^6q-1564385280p^7q+1411522560p^2q^2\\
&\qquad-38529715200p^3q^2+191572569600p^4q^2+234479232000p^5q^2+103600619520p^6q^2\\
&\qquad+166103040pq^3-17240448000p^2q^3+367911014400p^3q^3-2051089382400p^4q^3
\end{aligned}
$$

$$- 1568779100160p^5q^3 + 322560q^4 - 187852800pq^4 + 15351475200p^2q^4$$
$$- 543527731200p^3q^4 + 6127630540800p^4q^4 + 921600q^5 - 1062681600pq^5$$
$$+ 90862387200p^2q^5 - 1568779100160p^3q^5 + 1728000q^6 - 1944652800pq^6$$
$$+ 103600619520p^2q^6 + 2380800q^7 - 1564385280pq^7 + 1904640q^8\Big) + O(\tau^{12}). \tag{A5}$$

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